MATHEMATICS

UNIT 1

Sets:

A set is a collection of distinct objects, called elements, that are grouped together based on a common characteristic. Sets are denoted by enclosing the list of elements in curly braces {}. For example, the set of even numbers less than 10 can be denoted as {2, 4, 6, 8}.

Symbols in Sets:

- **∪ - Union**: The symbol ∪ is used to denote the union of two sets. For example, A ∪ B represents the set of all elements that belong to either A or B.
- **∩ - Intersection:** The symbol ∩ is used to denote the intersection of two sets. For example, A ∩ B represents the set of all elements that belong to both A and B.
- **⊆ - Subset:** The symbol ⊆ is used to denote that one set is a subset of another. For example, A ⊆ B means that every element of A is also an element of B.
- **⊂ - Proper subset**: The symbol ⊂ is used to denote that one set is a proper subset of another. For example, A ⊂ B means that every element of A is also an element of B, but there is at least one element in B that is not in A.
- **∈ - Element:** The symbol ∈ is used to denote that an element belongs to a set. For example, x ∈ A means that x is an element of the set A.
- **∅ - Empty set:** The symbol ∅ is used to denote the empty set, which is the set that has no elements.
- **\- Set difference:** The symbol \ is used to denote the set difference operation. For example, A \ B represents the set of all elements that belong to A but not to B.
- **| - Set-builder notation**: The symbol | is used in set-builder notation to separate the condition from the variable. For example, {x | x is an even integer} represents the set of all even integers.
- **= - Equality**: The symbol = is used to denote that two sets are equal. For example, A = B means that A and B have the same elements.
- **≠ - Inequality:** The symbol ≠ is used to denote that two sets are not equal. For example, A ≠ B means that A and B do not have the same elements.

Fundamental operations of sets:

Union:

The union of two sets is the set of all elements that are in either set. It is denoted by the symbol "∪". For example, suppose we have two sets A and B:

$A = \{1, 2, 3\}$

$B = \{3, 4, 5\}$

The union of A and B is:

$A \cup B = \{1, 2, 3, 4, 5\}$

In this case, the set {3} appears only once in the union because sets do not contain duplicates.

Intersection: The intersection of two sets is the set of all elements that are in both sets. It is denoted by the symbol "∩". For example, continuing with the sets A and B from above:

$A = \{1, 2, 3\}$

$B = \{3, 4, 5\}$

The intersection of A and B is:

$A \cap B = \{3\}$

This is because 3 is the only element that is in both sets.

Difference: The difference of two sets is the set of all elements that are in the first set but not in the second set. It is denoted by the symbol "−". For example, using the same sets A and B as above:

$A = \{1, 2, 3\}$

$B = \{3, 4, 5\}$

The difference of A and B is:

$A - B = \{1, 2\}$

This is because the elements 1 and 2 are in set A but not in set B.

Complement: The complement of a set is the set of all elements that are not in the set, but which belong to a larger set known as the universal set. The universal set is usually denoted by the symbol "U". For example, suppose we have a universal set U containing the integers from 1 to 6, and a set A containing the elements 2, 3, and 4:

$U = \{1, 2, 3, 4, 5, 6\}$

A = {2, 3, 4}

The complement of A with respect to U is:

A' = {1, 5, 6}

This is because the elements 1, 5, and 6 are not in A, but they are in the universal set U.

<u>Symmetric difference</u>: The symmetric difference of two sets is the set of all elements that are in either set but not in both. It is denoted by the symbol "Δ". For example, using the same sets A and B as above:

A = {1, 2, 3}

B = {3, 4, 5}

The symmetric difference of A and B is:

A Δ B = {1, 2, 4, 5}

This is because the elements 1, 2, 4, and 5 are in either set A or set B, but not in both sets.

The Principle of Inclusion and Exclusion

The Principle of Inclusion and Exclusion (PIE) is a counting technique used in combinatorics to determine the number of elements in the union of two or more sets, taking into account the overlaps and exclusions between them. The formula for PIE is as follows:

$|A1 \cup A2 \cup \ldots \cup An| = \sum|Ai| - \sum|Ai \cap Aj| + \sum|Ai \cap Aj \cap Ak| - \ldots + (-1)^{(n-1)}|A1 \cap A2 \cap \ldots \cap An|$

where |S| represents the cardinality (number of elements) of set S.

The formula includes the sums of the cardinalities of individual sets, the pairwise intersections, the triple intersections, and so on, with alternating signs (+ and -) for each subsequent term, to ensure that overlapping elements are not double-counted.

Here are three examples to illustrate the application of PIE:

<u>Example 1:</u> **There are 5 friends - Mohit Jangir, Samridhi Jha, Lokesh Sain, Ankit Kumawat, and Jay Bhati. They are members of different clubs in their school - the Music Club, the Drama Club, and the Photography Club. The number of members in each club is as follows: Mohit Jangir and Ankit Kumawat are in the Music Club and the Photography Club, Samridhi Jha is in the Drama Club and the Photography Club, Lokesh Sain is in the Drama Club only, and Jay Bhati is in all three clubs. How many students are in at least one of the clubs?**

Solution: Using PIE, we can find the number of students who are in at least one club, then subtract the number of students who are in exactly two clubs, and finally add back the number of students who are in all three clubs. We have:

|Music ∪ Drama ∪ Photography| = |Music| + |Drama| + |Photography| - |Music ∩ Drama| - |Music ∩ Photography| - |Drama ∩ Photography| + |Music ∩ Drama ∩ Photography|

Since Mohit Jangir and Ankit Kumawat are in both the Music and Photography Clubs, and Samridhi Jha is in both the Drama and Photography Clubs, and Jay Bhati is in all three clubs, we have:

|Music| = 2

|Drama| = 2

|Photography| = 4

and

|Music ∩ Photography| = |{Mohit Jangir, Ankit Kumawat}| = 2

|Drama ∩ Photography| = |{Samridhi Jha}| = 1

|Music ∩ Drama| = 0

|Music ∩ Drama ∩ Photography| = |{Jay Bhati}| = 1

Substituting these values into the PIE formula, we get:

|Music ∪ Drama ∪ Photography| = 2 + 2 + 4 - 0 - 2 - 1 + 1

= 6

Therefore, there are 6 students in at least one of the clubs.

Example 2: Suppose there are 100 students in a school, and 40 of them are in the choir, 30 of them are in the band, and 20 of them are in both. How many students are not involved in either activity?

Solution: Using PIE, we can find the number of students in the union of the two sets (choir and band) and then subtract it from the total number of students in the school. We have:

|Choir ⍰ Band| = |Choir| + |Band| - |Choir ∩ Band|

= 40 + 30 - 20

= 50

Therefore, the number of students not involved in either activity is:

|Not involved| = |School| - |Choir ⍰ Band|

= 100 - 50

= 50

Hence, there are 50 students not involved in either activity.

Example 3: A group of 50 people were asked about their preferred modes of transportation to work. 25 of them preferred driving, 20 of them preferred taking the train, and 15 of them preferred walking. 10 of them preferred driving and taking the train, 8 of them preferred driving and walking, and 5 of them preferred taking the train and walking. How many of them preferred all three modes of transportation?

Solution: Using PIE, we can find the number of people who preferred at least one mode of transportation and then subtract the number of people who preferred exactly two modes of transportation, before adding back the number of people who preferred all three modes of transportation. We have:

|Driving ∪ Train ∪ Walking| = |Driving| + |Train| + |Walking| - |Driving ∩ Train| - |Driving ∩ Walking| - |Train ∩ Walking| + |Driving ∩ Train ∩ Walking|

= 25 + 20 + 15 - 10 - 8 - 5 + |Driving ∩ Train ∩ Walking|

= 37 + |Driving ∩ Train ∩ Walking|

We are given that |Driving ∩ Train ∩ Walking| = x, where x is the number of people who preferred all three modes of transportation. Substituting this value into the equation above, we get:

|Driving ∪ Train ∪ Walking| = 37 + x

Since there are 50 people in the group, the number of people who preferred no mode of transportation is:

|No mode| = 50 - |Driving ∪ Train ∪ Walking|

= 50 - (37 + x)

= 13 - x

Therefore, there are 13 - x people who preferred no mode of transportation, and x people who preferred all three modes of transportation.

Principle of Mathematical Induction:

The principle of mathematical induction is a proof technique used to prove statements for all positive integers. It involves three steps: the base case, the inductive hypothesis, and the inductive step.

Step 1: Base Case

The first step in the principle of mathematical induction is to prove the statement for the first positive integer, usually 1. This is called the base case.

Step 2: Inductive Hypothesis

The second step is to assume that the statement is true for some arbitrary positive integer, k. This is called the inductive hypothesis.

Step 3: Inductive Step

The third step is to show that if the statement is true for k, then it must also be true for k + 1. This is called the inductive step.

If all three steps are satisfied, then the statement is true for all positive integers.

Example 1: Proving the sum of the first n positive integers is $n(n+1)/2$ using mathematical induction.

Base Case: When $n = 1$, the statement is true since the sum of the first positive integer is 1.

Inductive Hypothesis: Assume that the statement is true for some arbitrary positive integer, k. That is,

$1 + 2 + 3 + \ldots + k = k(k+1)/2$

Inductive Step: We want to show that if the statement is true for k, then it must also be true for k + 1. That is,

$1 + 2 + 3 + \ldots + k + (k+1) = (k+1)(k+2)/2$

Adding $(k+1)$ to both sides of the inductive hypothesis, we get:

$1 + 2 + 3 + \ldots + k + (k+1) = k(k+1)/2 + (k+1)$

Simplifying, we get:

$1 + 2 + 3 + \ldots + k + (k+1) = (k+1)(k+2)/2$

Therefore, the statement is true for all positive integers.

Example 2: Proving that $2^n > n^2$ for all $n \geq 5$ using mathematical induction.

Base Case: When $n = 5$, we have $2^5 = 32$ and $5^2 = 25$. Therefore, the statement is true for the base case.

Inductive Hypothesis: Assume that the statement is true for some arbitrary positive integer, k. That is,

$2^k > k^2$

Inductive Step: We want to show that if the statement is true for k, then it must also be true for k + 1. That is,

$2^{(k+1)} > (k+1)^2$

We can rewrite $2^{(k+1)}$ as $2*2^k$. Substituting this into the inequality, we get:

$2*2^k > (k+1)^2$

Simplifying, we get:

$2^k > (k+1)^2/4$

By the inductive hypothesis, we know that $2^k > k^2$. Therefore, we have:

$k^2 > (k+1)^2/4$

Expanding and simplifying, we get:

$3k^2 - 2k - 1 > 0$

This is true for all $k \geq 5$, so the statement is true for all positive integers.

Function:

A function is a rule that assigns to each element in a set X (called the domain) exactly one element in a set Y (called the range).

A function f can be represented using the following notation:

$f: X \to Y$

Where X is the domain, Y is the range, and → means "maps to". For example, if $f(x) = x^2$, then:

$f: \mathbb{R} \to \mathbb{R}$

This means that the domain of f is the set of all real numbers, and the range is also the set of all real numbers.

Example 1:

Let's consider the function $f(x) = 2x + 1$. The domain of this function is all real numbers, and the range is also all real numbers. To find the output of the function for a given input, we substitute the input value in the equation.

For example, if we want to find f(3), we substitute 3 for x in the equation:

$f(3) = 2(3) + 1 = 7$

Therefore, the output of the function for the input 3 is 7.

Example 2:Let's consider the function $g(x) = x^2 - 1$. The domain of this function is all real numbers, and the range is all non-negative real numbers. To find the output of the function for a given input, we substitute the input value in the equation.

For example, if we want to find g(-2), we substitute -2 for x in the equation:

$g(-2) = (-2)^2 - 1 = 3$

Therefore, the output of the function for the input -2 is 3.

Example 3:

Let's consider the function $h(x) = 1/x$. The domain of this function is all real numbers except for 0, and the range is all real numbers except for 0. To find the output of the function for a given input, we substitute the input value in the equation.

For example, if we want to find h(2), we substitute 2 for x in the equation:

$h(2) = 1/2$

Therefore, the output of the function for the input 2 is 1/2.

Relations

Relations refer to the way two or more sets of numbers or mathematical expressions are related to each other. A relation is a set of ordered pairs where each ordered pair consists of two elements that are related in some way. Relations can be represented in different ways, including graphs, tables, equations, and mappings.

Functions:

A function is a relation between two sets, where each element in the first set is related to exactly one element in the second set. Functions can be represented by equations, tables, or graphs. Examples of functions include:

a) $f(x) = x^2 – 1$: This is a quadratic function that relates every value of x to a corresponding value of y. For example, when x = 2, y = 3.

b) $g(x) = \sqrt{x}$: This is a square root function that relates every value of x to a unique non-negative value of y. For example, when x = 9, y = 3.

c) h(x) = sin(x): This is a trigonometric function that relates every value of x to a corresponding value of y between -1 and 1. For example, when $x = \pi/2$, $y = 1$.

Equivalence Relations:

An equivalence relation is a relation between two elements that satisfies three properties: reflexivity, symmetry, and transitivity. Equivalence relations can be represented by a table or a graph. Examples of equivalence relations include:

a) =: This is the equality relation that relates two elements that are the same. For example, 2 = 2.

b) ≡: This is the congruence relation that relates two elements that have the same remainder when divided by a fixed number. For example, $9 \equiv 3 \pmod 6$.

c) ~: This is the similarity relation that relates two elements that are similar in shape or form. For example, two triangles that have the same angles are similar.

Order Relations:

An order relation is a relation between two elements that specifies the order in which they occur. Order relations can be represented by a table or a graph. Examples of order relations include:

a) <: This is the less than relation that relates two elements where one element is smaller than the other. For example, $3 < 5$.

b) >: This is the greater than relation that relates two elements where one element is larger than the other. For example, $7 > 4$.

c) ≤: This is the less than or equal to relation that relates two elements where one element is smaller than or equal to the other. For example, $4 \leq 4$.

Equivalence Relation

In mathematics, an equivalence relation is a relation that satisfies the following three properties: reflexivity, symmetry, and transitivity. An equivalence relation divides a set into distinct classes, where all elements within a class are related to each other, but elements in different classes are not related to each other. Equivalence relations have a variety of applications, including in group theory, topology, and graph theory. In this article, we will discuss equivalence relations in detail, including their properties and examples.

Properties of Equivalence Relation:

- Reflexivity: A relation R is said to be reflexive if for every element a belongs to A, (a,a) is a part of the relation R.
- Symmetry: A relation R is said to be symmetric if for every (a,b) belongs to R, (b,a) is also a part of R.

- Transitivity: A relation R is said to be transitive if for every (a,b) and (b,c) belongs to R, (a,c) is also a part of R.

Example 1:

Let A be a set of integers. Define the relation R on A as follows: (a,b) belongs to R if and only if a and b have the same remainder when divided by 3.

We can check that R is an equivalence relation on A.

Reflexivity: Every integer has the same remainder as itself when divided by 3. So, (a,a) belongs to R for every a belongs to A.

Symmetry: If a and b have the same remainder when divided by 3, then b and a also have the same remainder when divided by 3. So, if (a,b) belongs to R, then (b,a) belongs to R.

Transitivity: If a, b, and c have the same remainder when divided by 3, then a and c also have the same remainder when divided by 3. So, if (a,b) and (b,c) belongs to R, then (a,c) also belongs to R.

Example 2:

Let A be a set of people. Define the relation R on A as follows: (a,b) belongs to R if and only if a and b have the same last name.

We can check that R is an equivalence relation on A.

Reflexivity: Every person has the same last name as themselves. So, (a,a) belongs to R for every a belongs to A.

Symmetry: If a and b have the same last name, then b and a also have the same last name. So, if (a,b) belongs to R, then (b,a) belongs to R.

Transitivity: If a, b, and c have the same last name, then a and c also have the same last name. So, if (a,b) and (b,c) belongs to R, then (a,c) also belongs to R.

Example 3:

Let A be a set of integers. Define the relation R on A as follows: (a,b) belongs to R if and only if a-b is a multiple of 5.

We can check that R is an equivalence relation on A.

Reflexivity: Every integer is a multiple of itself. So, (a,a) belongs to R for every a belongs to A.

Symmetry: If a-b is a multiple of 5, then b-a is also a multiple of 5. So, if (a,b) belongs to R, then (b,a) belongs to R.

Transitivity: If a-b is a multiple of 5 and b-c is a multiple of 5, then a-c is also a multiple of 5. So, if (a,b) and (b,c) belongs to R.

Partition

In set theory, a partition of a set is a collection of non-empty, mutually disjoint subsets of the original set that cover the entire set. In other words, each element of the original set belongs to exactly one of the subsets in the partition. The subsets in the partition are called the blocks or parts of the partition.

Formally, a partition P of a set X is a set of non-empty subsets of X, denoted {B1, B2, ..., Bn}, such that:

The union of all the subsets in P equals X:

B1 U B2 U ... U Bn = X

The subsets are pairwise disjoint:

Bi ∩ Bj = Ø for i ≠ j

Each subset is non-empty:

Bi ≠ Ø for all i

Example 1:

Consider the set X = {1, 2, 3, 4, 5, 6, 7, 8} and the partition P = {{1, 2}, {3, 4, 5}, {6, 7, 8}}. The partition P satisfies the above conditions since the union of the subsets is X, the subsets are pairwise disjoint, and each subset is non-empty.

Example 2:

Let X = {a, b, c, d, e} and P = {{a, b}, {c}, {d, e}} be a partition of X. Then, the subsets in the partition satisfy the conditions since the union of the subsets is X, the subsets are pairwise disjoint, and each subset is non-empty.

Example 3:

Consider the set of all integers Z and the partition P = {{n ∈ Z | n is even}, {n ∈ Z | n is odd}}. This is a partition of Z since every integer is either even or odd, and no integer can be both at the same time. The subsets are also pairwise disjoint and each subset is non-empty.

Partitions are useful in many areas of mathematics, including group theory, topology, and combinatorics. They can be used to define equivalence relations and to count objects in combinatorial problems.